EULER ELASTIKASI NOAN’ANAVIY ELASTIKLIK NAZARIYASIDA

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Annotatsiya: Mazkur ishda Euler–Bernoulli nazariyasiga asoslangan elastik chiziq modeli hamda uning nolo-kal (Eringen tipidagi) modifikatsiyasi ko‘rib chiqiladi. Lokal elastiklik modeli doirasida egilish momenti va egri-lik o‘rtasidagi bog‘lanishdan foydalanib, Euler elastikasi uchun bosh differensial tenglama keltirib chiqariladi. Ushbu nochiziqli ikkinchi tartibli oddiy differensial tenglama bir marta integrallash orqali birinchi integralga keltiriladi va Yakobi elliptik funksiyalari hamda birinchi va ikkinchi turdagi to‘liq bo‘lmagan elliptik integrallar yordamida aniq analitik yechim olinadi. Keyinchalik kichik o‘lchamli effektlarni hisobga oluvchi Eringen nolo-kal elastiklik nazariyasi asosida konstitutiv bog‘lanish modifikatsiya qilinadi. Natijada lokal modelning umumlashgan ko‘rinishi olinadi va yangi bosh tenglama chiqariladi. Ushbu tenglama ham integrallash yo‘li bilan elliptik integrallar orqali bilvosita parametrik ko‘rinishda ifodalanadi. Olingan natijalar nano-balkalar va nanotuzilmalarning deformatsiyasini aniqlashda qo‘llanilishi mumkin.

Kalit so‘zlar: Euler elastikasi, Leonhard Euler nazariyasi, Daniel Bernoulli balkasi, lokal elastiklik, nolo-kal elastiklik, egilish momenti, egri-lik, Yakobi elliptik funksiyalari, to‘liq bo‘lmagan elliptik integrallar, mexanik muvozanat

Abstract: This work examines the elastic line model based on the Euler–Bernoulli theory and its nonlocal (Eringen-type) modification. Within the local elasticity model, by employing the relationship between the bending moment and curvature, the governing differential equation for Euler elasticity is derived. This nonlinear second-order ordinary differential equation is reduced to a first integral by a single integration, and an exact analytical solution is obtained using Jacobi elliptic functions and elliptic integrals of the first and second kind. Subsequently, the constitutive relation is modified based on Eringen’s nonlocal elasticity theory that accounts for small-scale effects. As a result, a generalized form of the local model is obtained and a new governing equation is derived. This equation is also expressed in a parametric form via elliptic integrals by integration. The obtained results can be used to determine the deformation of nano-beams and nanostructures.

Keywords: Euler beam, Leonhard Euler’s theory, Daniel Bernoulli beam, local elasticity, nonlocal elasticity, bending moment, curvature, Jacobi elliptic functions, incomplete elliptic integrals, mechanical equilibrium

Аннотация: В данной статье рассматривается модель упругой линии, основанная на теории Эйлера-Бернулли, и ее нелокальная (типа Эрингена) модификация. В рамках локальной модели упругости, используя соотношение между изгибающим моментом и кривизной, выводится основное дифференциальное уравнение для упругости Эйлера. Это нелинейное обыкновенное дифференциальное уравнение второго порядка сводится к первому интегралу с помощью одного интегрирования, и получается точное аналитическое решение с использованием эллиптических функций Якоби и эллиптических интегралов первого и второго рода. Затем конститутивное соотношение модифицируется на основе нелокальной теории упругости Эрингена, которая учитывает мелкомасштабные эффекты.

В результате получается обобщенная форма локальной модели и выводится новое определяющее уравнение. Это уравнение также выражается в параметрической форме, неявно через эллиптические интегралы посредством интегрирования. Полученные результаты могут быть использованы для определения деформации нанобалок и наноструктур.

Ключевые слова: балка Эйлера, теория Леонарда Эйлера, балка Даниэля Бернулли, локальная упругость, нелокальная упругость, изгибающий момент, кривизна, эллиптические функции Якоби, неполные эллиптические интегралы, механическое равновесие

Klassik Eyler–Bernoulli modeli XVIII asrda mahalliy elastiklik nazariyasiga asoslanib tekis egri shaklli shinalarning katta deformatsiyalarini tasvirlash maqsadida ishlab chiqilgan. Ko‘rsatish mumkinki, bu muammoning aniq yechimi Yakobi elliptik funksiyalari va birinchi, ikkinchi hamda uchinchi turdagi to‘liq bo‘lmagan elliptik integralardan foydalangan holda yozilishi mumkin.

Euler-Bernoulli nazariyasi juda kichik ob’ektlarga, masalan, nano-balka (jumladan nano-simlar [5], nanotubalar, nanotirsaklar va boshqalar) ga ham qo‘llanilishi mumkin; ular kichik uzunlik o‘lchoviga (nano o‘lcham) ega bo‘lgan va egilish momentlari, shuningdek kesish hamda o‘q bo‘yicha kuchlar ta’sirida deformatsiyalanadigan balka sifatida ko‘riladi. Nanokuchlamadagi bunday ob’ektlar g‘ayrioddiy fizik va mexanik xususiyatlarga ega: yuqori uzunlik-kenglik nisbati, yuqori egiluvchanlik, yuqori tortish va kesish mustahkamligi hamda yuqori elastiklik moduli. Masalan, uglerod nanotubalar uchun

$$\rho = 2300\frac{kg}{m^3}, \qquad E = 1000GPa, \qquad \nu = 0.19, \qquad G = 420GPa$$

$$d = 1.0nm, \qquad A = 0.785nm^2, \qquad I = \frac{\pi d^2}{64} = 0.0491nm^4, \qquad l_i = 1.5nm \quad (1)$$

bu yerda ρ zichlik, E Yong (elastik) moduli, ν Poisson nisbat, G Kirxgof (kesish) moduli, d nanotubalar diametri, A ularning kesim maydoni, I ularning kesim inersiya momenti va l_i ichki xarakteristik uzunlikdir.

Nanotirsaklarni tasvirlash uchun nafaqat klassik tirgak nazariyasi, balki kichik o‘lchamli ta’sirlarga oid nojoyi elastiklik ham qo‘llaniladi, bu esa masalan, statik egilish, erkin tebranish tahlili va uglerod nanotubalarining elastik bukilish muammolarini o‘rganishga imkon beradi.

Lokal elastika uchun aniq analitik yechimlar

Elastik chiziq yoki Euler elastikasi formal matematik tavsifi elliptik funksiyalar yordamida umumiy yechimlar hamda erkin elastika va tortish kuchiga ega elastika uchun aniq parametrizatsiyalar taqdim etilgan. Elastiklik nazariyasida nurning kichik segmenti uchun muvozanat tenglamalari quyidagicha beriladi:

$$\frac{dM}{ds} = Q, \qquad \frac{dQ}{ds} = -\kappa N, \qquad \frac{dN}{ds} = \kappa Q \qquad (2)$$

bu yerda M egilish momenti, N va Q esa o‘q bo‘ylab kuch va kesish kuchi, va κ elastikaning egri chiziqligi bo‘lib, u shunday deyiladigan arka uzunligi parametri s bilan parametrlangan,s 0 dan L gacha o‘zgaradi, bu yerda L elastikaning umumiy uzunligi (ba’zan, umumiylikni yo‘qotmasdan, birlik elastika ko‘rib chiqiladi, shuning uchun $L = 1$).

Agar $\theta(s)$ni kompressiv yuk F ham x o‘qi bo‘ylab yo‘naltirilgan bo‘lgan va x o‘qi bo‘ylab joylashgan elastik chiziqning istalgan nuqtasi $P\big(x(s), y(s)\big)$dagi tegish burchagi deb aniqlasak, unda Euler elastik chizig‘ining bukilishi (masalan, kesish bo‘ylama egilishi) y o‘qi bo‘ylab sodir bo‘ladi. Bunday vaziyatda o‘q bo‘ylab kuch va kesish kuchi, egri chiziqlik va geometrik sharoitlar quyidagicha beriladi:

$$N = -F\cos\theta\,, \qquad Q = F\sin\theta\,, \qquad \frac{d\theta}{ds} = \kappa, \qquad \frac{dx}{ds} = \cos\theta\,, \qquad \frac{dy}{ds} = \sin\theta \quad (3)$$

Klassik (lokal) nur modeli bo‘yicha elastika har qanday nuqtasidagi egilish momenti uning egri chiziqligiga proporsional bo‘ladi, ya’ni egilish momenti va egri chiziqlik o‘rtasidagi munosabat Euler-Bernulli qonuni bilan ifodalanadi:

$$M = -EI\kappa = -EI\frac{d\theta}{ds} \qquad (4)$$

egiluvchan qattiqlik EI Yung moduli E va inersiya momenti I orqali ifodalanadi.

Bundan tashqari, [2] da ko‘rsatilishicha, egilish momenti M va elastik egri bo‘ylab ta’sir etuvchi aksial kuch N tenglama orqali bog‘langan

$$\frac{M^2}{2EI} = |N| \qquad (5)$$

bu mexanik muvozanat holatida elastika bo‘ylab har bir nuqtada kuchlar yig‘indisi nolga teng bo‘lishi talabining ifodasi deb tushunilishi mumkin.

(2)-(4) dan foydalangan holda bosh tenglamani quyidagicha aniqlash mumkin:

$$\frac{d^2\theta}{ds^2} = -\frac{1}{EI}\frac{dM}{ds} = -\frac{Q}{EI} = -\frac{F}{EI}\sin\theta = -\alpha^2\sin\theta\,, \qquad \alpha^2 = \frac{F}{EI} \qquad (6)$$

Yuqoridagi hosilaga nisbatan yechilmagan chiziqli ikkinchi tartibli oddiy differensial tenglama $\theta(s)$ funksiyasi bo‘yicha (7)-ning chap va o‘ng tomonlarini $2\frac{\theta(s)}{ds}$ ga ko‘paytirib bir marta integrallash mumkin. Shunda quyidagini olamiz:

$$\frac{d}{ds}\left(\left(\frac{d\theta}{ds}\right)^2\right) = 2\alpha^2\frac{d}{ds}(\cos\theta) \qquad (7)$$

Buni quyidagicha integrallash mumkin:

$$\frac{d\theta}{ds} = \pm\sqrt{2\alpha^2\cos\theta + C} = \frac{2\alpha}{k}\sqrt{1 - k^2\sin^2\left(\frac{\theta}{2}\right)}, \qquad k = \frac{4\alpha^2}{C+2\alpha^2} \qquad (8)$$

bu yerda C , yoki ekvivalent ravishda k ($\alpha \neq 0$, ya’ni $F \neq 0$ bo‘lganda), birinchi integral konstantasidir.

Ta’kidlash joizki, integral konstantasi C o‘lchamsiz emas, balki nm^{-2} o‘lchamga ega, ya’ni α^2 atamasining o‘lchami bilan bir xil. $k^2 \leq 1$ bo‘lgan holatda (ya’ni $C \geq 2\alpha^2$), yuqoridagi ifoda quyidagicha integrallanadi:

$$s(\theta) = \pm\frac{k}{2\alpha}\int\frac{d\theta}{\sqrt{1-k^2\sin^2\left(\frac{\theta}{2}\right)}} = s_0 \pm \frac{k}{2\alpha}F\left(\frac{\theta}{2}, k\right) \qquad (9)$$

Bu yerda s_0 ikkinchi integral konstantasi va ushbu

$$F(\varphi, k) = \int_0^{\varphi}\frac{d\varphi}{\sqrt{1-k^2\sin^2\varphi}} \qquad (10)$$

funksiya birinchi tur elliptik integraldir.

Shunday qilib, (9) ifodani teskari qilganimizda, ya’ni elliptik sinus funksiyasi $sn(u,k) = \sin\phi$ birinchi turdagi to‘liq bo‘lmagan elliptik integral $F(\phi,k) = u$ ning oddiy teskari funksiyasi ekanligini hisobga olsak, (6) ning umumiy yechimini quyidagicha yozish mumkin:

$$\theta(s) = \pm 2\arcsin\left(sn\left(\frac{\alpha}{k}(s-s_0), k\right)\right), \qquad \frac{d\theta}{ds} = \pm\frac{2\alpha}{k}dn\left(\frac{\alpha}{k}(s-s_0), k\right) \qquad (11)$$

bu yerda biz (8) ni (11) ko‘rinishida qayta yozish uchun Jacobi elliptik delta va sinus funksiyalari o‘rtasidagi bog‘lanishdan foydalandik, ya’ni,

$$dn^2(u,k) = 1 - u^2sn^2(u,k) = 1 - k^2\sin^2\varphi \qquad (12)$$

Ikkala integral konstantalari, ya’ni k (yoki C) va s_0, egri nurning chegara qiymat masalalari (BVPs) uchun tegishli chegara sharoitlari (mahkam ushlab turilgan, oddiy supportlangan va hokazo) qo‘llanilishi orqali aniqlanishi mumkin. (11) dan foydalangan holda quyidagini olishimiz mumkin:

$$\sin\theta = \pm 2sn\left(\frac{\alpha}{k}(s-s_0), k\right)cn\left(\frac{\alpha}{k}(s-s_0), k\right)$$

$$\cos\theta = 1 - 2sn^2\left(\frac{\alpha}{k}(s-s_0), k\right) \qquad (13)$$

bu yerda biz Jacobi elliptik kosinus va sinus funksiyalari o‘rtasidagi bog‘lanishdan foydalanganmiz, ya’ni,

$$cn^2(u,k) = 1 - sn^2(u,k) = 1 - \sin^2\varphi = \cos^2\varphi \qquad (14)$$

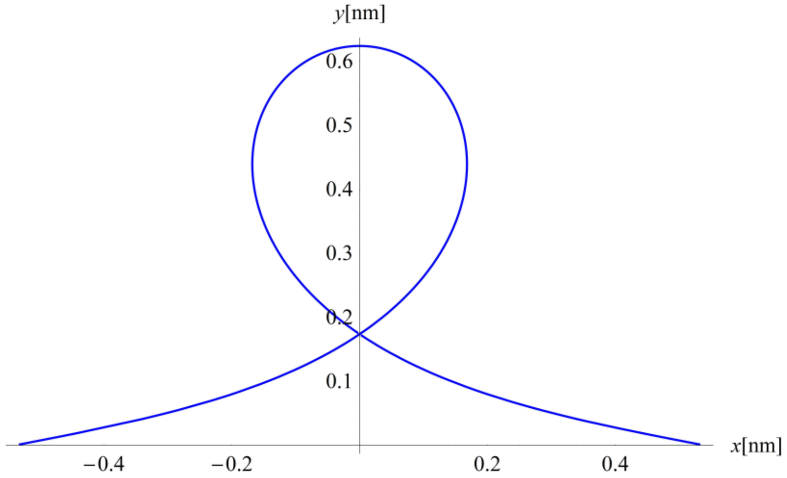
Keyin (3)-dagi oxirgi ikki tenglamani birlashtirib va (8)-dan foydalangan holda, Euler elastikasidagi istalgan nuqtaning koordinatalarini olamiz (olingan umumiy yechimning namunaviy grafigi uchun 1-rasmga qarang), ya’ni,

$$x(\theta) = \int \cos\theta\, ds(\theta) = \pm\frac{k}{2\alpha}\int \frac{1-2\sin^2\theta}{\sqrt{1-k^2\sin^2\theta}}d\theta = x_0 \pm \frac{k}{\alpha}\left(1-\frac{2}{k^2}\right)F\left(\frac{\theta}{2},k\right) \pm \frac{2}{\alpha k}E\left(\frac{\theta}{2},k\right) \quad (15)$$

$$y(\theta) = \int \sin\theta\, ds(\theta) = \mp\frac{1}{2\alpha^2}\int \frac{d\left(2\alpha^2\cos(\theta)+C\right)}{\sqrt{2\alpha^2\cos(\theta)+C}} = y_0 \mp \frac{1}{\alpha^2}\sqrt{2\alpha^2\cos(\theta)+C} = y_0 \mp \frac{2}{\alpha k}\sqrt{1-k^2\sin^2\left(\frac{\theta}{2}\right)} \quad (16)$$

Bu yerda $x_0,\ y_0$ integral konstantalari

$$E(\varphi,k) = \int_0^{\varphi} \sqrt{1-k^2\sin^2\varphi}\, d\varphi \qquad (17)$$



1-rasm: (15) va (16) orqali aniqlangan parametrik yechimning $\left(x(\theta),y(\theta)\right)$ namunaviy grafigi, integral konstantalari $x_0 = y_0 = 0$, $C = 20nm^{-2}$, elliptik modul $\frac{1}{k} = 0.99549$, kuch $F = 500nN$, Yong modul $E = 1000GPa$, inertiya momenti $I = 0.0491nm^4$.

— ikkinchi turdagi to‘liq bo‘lmagan elliptik integral.

(11)-dan foydalangan holda (15)–(16) ifodalari shuningdek s funksiyalari sifatida ham yozilishi mumkin, ya’ni,

$$x(s) = x_0 + \left(1-\frac{2}{k^2}\right)(s-s_0) \pm \frac{2}{\alpha k}E\left(am\left(\frac{\alpha}{k}(s-s_0),k\right),k\right)$$

$$y(s) = y_0 \mp \frac{2}{\alpha k}dn\left(\frac{\alpha}{k}(s-s_0),k\right) \qquad (18)$$

Bu yerda $am(u,k) = \phi$ Jacobi elliptik amplitudasi.

Xuddi shunday, $k^2 > 1$ (ya’ni, $|C| < 2\alpha^2$) bo‘lgan holatda, transformatsiyani qo‘llash orqali

$$F(\varphi,k) = \frac{1}{k}F\left(\arcsin\left(k\sin\left(\frac{\theta}{2}\right)\right),\frac{1}{k}\right) \qquad (19)$$

Biz (9) dagi umumiy yechimni quyidagicha yozish mumkinligini aniqladik:

$$s(\theta) = s_0 \pm \frac{1}{\alpha}F\left(\arcsin\left(k\sin\left(\frac{\theta}{2}\right)\right),\frac{1}{k}\right) \qquad (20)$$

(20) ifodani teskari ko‘tarib, [4] ni olamiz:

$$\theta(s) = \pm 2\arcsin\left(\frac{1}{k}sn\left(\alpha(s-s_0),\frac{1}{k}\right)\right), \qquad \frac{d\theta}{ds} = \pm\frac{2\alpha}{k}cn\left(\alpha(s-s_0),\frac{1}{k}\right) \qquad (21)$$

Shuning uchun,

$$\sin\theta = \pm\frac{2}{k}sn\left(\alpha(s-s_0),\frac{1}{k}\right)dn\left(\alpha(s-s_0),\frac{1}{k}\right)$$

$$\cos\theta = 1-\frac{2}{k^2}sn^2\left(\alpha(s-s_0),\frac{1}{k}\right) \qquad (22)$$

Nihoyat, (3) dagi oxirgi ikki tenglamani birlashtirib va (8) ni qo‘llab, deformatsiyalangan shinali Euler elastikasidagi istalgan nuqtaning koordinatalarini quyidagicha aniqlaymiz:

$$x(s) = \int \cos\theta\, ds = s - s_0 - \frac{2}{k^2}\int sn^2\left(\alpha(s-s_0),\frac{1}{k}\right)ds = x_0 - s + s_0 \pm \frac{2}{\alpha}E\left(am\left(\alpha(s-s_0),\frac{1}{k}\right),\frac{1}{k}\right) (23)$$

$$y(s) = \int \sin\theta\, ds = \pm\frac{2}{k}\int sn\left(\alpha(s-s_0),\frac{1}{k}\right) dn\left(\alpha(s-s_0),\frac{1}{k}\right) ds = y_0 \mp \frac{2}{\alpha k} cn\left(\alpha(s-s_0),\frac{1}{k}\right) \quad (24)$$

Nolokal elastika uchun o‘yin differensial modeli

Kichik miqyosli ta’sirlar nurlar yoki nano-nurlarning normal bosim va deformatsiyasi o‘rtasidagi konstitutiv munosabatlarning tegishli o‘zgartirishlari (integral yoki differensial shakllarda) yordamida klassik (joylik) elastiklik nazariyasi bilan birlashtirilishi mumkin. Eringen yondashuviga ko‘ra (masalan, [3] ga qarang), biz umumiy stress-deformatsiya munosabatining integral ko‘rinishi orqali chiziqli elastiklik nazariyasiga shunday deb ataladigan deformatsiya tomonidan boshqariladigan nojoylikni kiritishimiz mumkin.

$$\sigma_{ij}^{kl}(\vec{x}) = \int k_\mu\left(\left|\vec{x}-\overrightarrow{x'}\right|\right)\sigma_{ij}^{l}\left(\overrightarrow{x'}\right)dV' = E\int k_\mu\left(\left|\vec{x}-\overrightarrow{x'}\right|\right)\varepsilon_{ij}\left(\overrightarrow{x'}\right)dV' \qquad (25)$$

Bu yerda $\sigma_{ij}^{kl}(x)$ - x nuqtadagi nolokal kuchlanishni ifodalaydi, $\sigma_{ij}^{l}(x') = E\varepsilon_{ij}(x')$ esa x' **r**eferens nuqtada lokal (va chiziqli) kuchlanish–deformatsiya munosabatini aniqlaydi. $k_\mu(|x-x'|)$ — bu so‘nuvchi funksiya bo‘lib, u nurlar yoki nanonurlardagi uzoq masofali o‘zaro ta’sirlarni ifodalaydi. Ushbu yadro (kernel) funksiya masshtablovchi koeffitsient $\mu = \varepsilon_0^2 l_i^2$ ga bog‘liq bo‘lib, u nolokallik darajasini tavsiflovchi parametr sifatida tushuniladi (o‘lchov birligi — uzunlikning kvadrati).

Umumiy holda, u material parametri ε_0 va ichki xarakteristik uzunlik l_i ga bog‘liq funksiya sifatida aniqlanadi (masalan, panjara parametri, granulalar o‘lchami, C–C bog‘lari orasidagi masofa va hokazo) [8, 9].

Eringenning nolokal nazariyasida biz yuqoridagi nolokal yadro $k_\mu(|x-x'|)$ ni ma’lum bir chiziqli differensial operator L_μ uchun Grin funksiyasi deb qabul qilamiz, shundayki

$$L_\mu k_\mu\left(\left|\vec{x}-\overrightarrow{x'}\right|\right) = \delta\left(\left|\vec{x}-\overrightarrow{x'}\right|\right) \qquad (26)$$

Yuqoridagi differensial operatorning eng keng tarqalgan tanlovi $L_\mu = 1-\mu\nabla^2$ bo‘lib, bu yerda ∇^2 Laplas operatoridir. Bu shuni anglatadiki, (25) da berilgan stress-strain konstitutiv munosabatining integral shaklini mos differensial shaklga ekvivalent tarzda qayta yozishimiz mumkin (masalan, [8, 9] ga qarang):

$$L_\mu\sigma_{ij}^{nl} = \sigma_{ij}^{nl} - \mu\nabla^2\sigma_{ij}^{nl} = \sigma_{ij}^{l} = E\varepsilon_{ij} \qquad (27)$$

Ushbu maqolada biz no-lokal Euler elastikalarini tasvirlash uchun oddiy differensial modelni ko‘rib chiqamiz, unda fazoviy o‘zgaruvchi $x(s)$ (s arka uzunligi parametri bo‘yicha parametrlangan) nisbatan Laplasian o‘rniga, ya’ni $\nabla_{x(s)}^2$ biz material o‘zgaruvchi s (ark uzunlik parametri o‘zi) bo‘yicha Laplasian, ya’ni ∇_s^2 ni ishlatamiz.

Keyin, bizning holatimizda, no-lokal stress-strain munosabati quyidagi ko‘rinishga ega:

$$\sigma_{xx} - \mu\cdot\frac{d^2\sigma_{xx}}{ds^2} = E\varepsilon_{xx} \qquad (28)$$

Bu yerda normal (x o‘qi bo‘ylab) no-lokal kuchlanish σ_{xx}, s arka uzunligi parametriga ega $P(s)$ nuqtada hisoblanganda, nafaqat shu nuqtadagi normal cho‘zilish ε_{xx} ga, balki Euler elastik chizig‘ining boshqa barcha nuqtalaridagi normal cho‘zilish qiymatlariga ham bog‘liq (boshqacha qilib aytganda, bizda shunday deyiladigan cho‘zilishga asoslangan no-lokal bo‘lish mavjud). Yuqoridagi (28) noan’anaviy konstitutiv bog‘lanishni Euler-Bernoulli балка nazariyasiga qo‘llash egilish momenti-egri-lik bog‘lanishi (4) ni o‘zgartirishga olib keladi, bu o‘zgarish esa bizning oddiy differensial modelimiz uchun endi quyidagicha ifodalanadi:

$$M - \mu\frac{d^2M}{ds^2} = -EI\kappa = -EI\frac{d\theta}{ds} \qquad (29)$$

So‘ng (2) va (3) dan quyidagini keltirib chiqaramiz:

$$\frac{d^2M}{ds^2} = \frac{dQ}{ds} = -\kappa N = F\cos\theta\cdot\frac{d\theta}{ds} \qquad (30)$$

(30)ni (29)ga qo‘yib hisoblasak, quyidagini olamiz:

$$M = -EI(1-\mu\alpha^2\cos\theta)\frac{d\theta}{ds}, \qquad \alpha^2 = \frac{F}{EI} \qquad (31)$$

bu µ nolga yaqinlashganda (5)-ga kamaytirilishi mumkin. (31)-ni arc uzunligi parametri s bo‘yicha differensiallab va yana (2)-(3)-ni qo‘llagan holda, bosh tenglama (6)-ning quyidagi o‘zgartirilgan ko‘rinishini olamiz:

$$(1-\mu\alpha^2\cos\theta)\frac{d^2\theta}{ds^2}+\mu\alpha^2\sin^2\theta\left(\frac{d\theta}{ds}\right)=-\alpha^2\sin\theta \qquad (32)$$

Agar biz shuni aniqlasak

$$P(\theta)=1-\mu\alpha^2\cos\theta=1-\mu\alpha^2+2\mu\alpha^2\sin^2\frac{\theta}{2} \qquad (33)$$

Keyin (32) quyidagicha yozilishi mumkin

$$P(\theta)\cdot\frac{d^2\theta}{ds^2}+P'(\theta)\left(\frac{d\theta}{ds}\right)^2=-\frac{1}{\mu}P'(\theta) \qquad (34)$$

Yuqoridagi nochiziqli ikkinchi tartibli oddiy differensial tenglama θ(s) funksiyasi bo‘yicha (34) ning chap va o‘ng tomonlarini $2P(\theta)\left(\frac{d\theta}{ds}\right)$ ga ko‘paytirib bir marta integrallash mumkin. Shunda quyidagini olamiz:

$$\frac{d}{ds}\left(P^2(\theta)\left(\frac{d\theta}{ds}\right)^2\right)=-\frac{1}{\mu}\frac{d}{ds}\left(P^2(\theta)\right) \qquad (35)$$

Buni quyidagicha integrallash mumkin:

$$\left(1+\mu\left(\frac{d\theta}{ds}\right)^2\right)P^2(\theta)=D^2>0 \qquad (36)$$

Bundan quyidagi natijani olishimiz mumkin:

$$\frac{d\theta}{ds}=\pm\sqrt{\frac{1}{\mu}\left(\frac{D^2}{P^2(\theta)}-1\right)}=\pm\frac{K}{\mu}\cdot\frac{\sqrt{\left(1-k_1\sin^2\frac{\theta}{2}\right)\left(1-k_2\sin^2\frac{\theta}{2}\right)}}{1-k_3\sin^2\frac{\theta}{2}} \qquad (37)$$

bu yerda D birinchi integral konstantasi va

$$k_1=\frac{2\mu\alpha^2}{\mu\alpha^2-1+D}, \qquad k_2=\frac{2\mu\alpha^2}{\mu\alpha^2-1-D}, \qquad k_3=\frac{2\mu\alpha^2}{\mu\alpha^2-1} \qquad (38)$$

Qiziqarli faktni qayd etaylik: (32) ifodasi µ nolga yaqinlashganda (6) ga osonlikcha qayta keltirilishi mumkin bo‘lsa-da, (37) va (8) o‘rtasidagi bog‘liqlik shunchalik ravshan emas (0 ga bo‘linish tufayli). (37)ni (34)ga qo‘yib, θ(s)ning ikkinchi hosilasi uchun ixcham ifodani ham olishimiz mumkin, ya’ni,

$$\frac{d^2\theta}{ds^2}=-\frac{D^2}{\mu}\cdot\frac{P'(\theta)}{P^3(\theta)}=\frac{D^2\alpha^2\sin\theta}{(\mu\alpha^2\cos\theta-1)^3}. \qquad (39)$$

Agar (37) dagi o‘zgaruvchilarni ajratsak va olingan ifodalarga integral olsak, unda

$$s(\theta)=\pm\frac{\sqrt{\mu}}{K}\int\frac{\left(1-k_3\sin^2\frac{\theta}{2}\right)d\theta}{\sqrt{\left(1-k_1\sin^2\frac{\theta}{2}\right)\left(1-k_2\sin^2\frac{\theta}{2}\right)}} \qquad (40)$$

bu esa bizni deformatsiyalangan elastika koordinatalarining mos ifodalariga ham olib keladi, ya’ni

$$x(\theta)=\int\cos\theta\,ds(\theta)=s(\theta)\mp\frac{2\sqrt{\mu}}{K}\int\frac{\left(1-k_3\sin^2\frac{\theta}{2}\right)\sin^2\frac{\theta}{2}\,d\theta}{\sqrt{\left(1-k_1\sin^2\frac{\theta}{2}\right)\left(1-k_2\sin^2\frac{\theta}{2}\right)}}$$

$$y(\theta)=\int\sin\theta\,ds(\theta)=\pm\frac{4\sqrt{\mu}}{K}\int\frac{\left(1-k_3\sin^2\frac{\theta}{2}\right)\sin\frac{\theta}{2}d\left(\sin\frac{\theta}{2}\right)}{\sqrt{\left(1-k_1\sin^2\frac{\theta}{2}\right)\left(1-k_2\sin^2\frac{\theta}{2}\right)}} \qquad (41)$$

(41)-dagi ikkinchi ifoda osonlik bilan integrallanishi mumkinligini ko‘raylik, ya’ni quyidagini olamiz:

$$y(\theta)=y_0\mp2\cdot\sqrt{\frac{\mu}{k_1k_2}}\cdot\sqrt{\left(1-k_1\sin^2\frac{\theta}{2}\right)\left(1-k_2\sin^2\frac{\theta}{2}\right)} \qquad (42)$$

By yerda y_0 - integral konstantasi.

To‘liq bo‘lmagan elliptik integrallar orqali bilvosita parametrizatsiya

$\tan\left(\frac{\theta}{2}\right)=t$ belgilash kiritib quyidagilarni olamiz:

$$\sin\theta = \frac{2t}{1+t^2}, \qquad \cos\theta = \frac{1-t^2}{1+t^2}, \qquad d\theta = \frac{2dt}{1+t^2} \qquad (43)$$

Shunda (40)-ifoda va (41) dagi birinchi ifoda quyidagi koʻrinishga keltiriladi:

$$s(t) = \pm\frac{2\sqrt{\mu}}{KK'}\int\frac{Q_2(t)dt}{(1+t^2)\sqrt{P_4(t)}}$$

$$x(t) = \pm\frac{2\sqrt{\mu}}{KK'}\int\frac{Q_4(t)dt}{(1+t^2)^2\cdot\sqrt{P_4(t)}} \qquad (44)$$

bu yerda $Q_2(t)$, $Q_4(t)$ va $P_4(t)$ koʻphadlar quyidagicha aniqlanadi:

$$Q_2(t) = k_3' - t^2, \qquad Q_4(t) = k_3' - k_3k_3't^2 + t^4, \qquad P_4(t) = (t^2 - k_1')(t^2 - k_2')$$

(45)

koeffitsientlar esa quyidagicha beriladi:

$$k_1' = \frac{1}{k_1 - 1} = \frac{\mu\alpha^2 - 1 + D}{\mu\alpha^2 + 1 - D}, \qquad k_1k_1' = \frac{2\mu\alpha^2}{\mu\alpha^2 + 1 - D},$$

$$k_2' = \frac{1}{k_2 - 1} = \frac{\mu\alpha^2 - 1 - D}{\mu\alpha^2 + 1 + D}, \qquad k_2k_2' = \frac{2\mu\alpha^2}{\mu\alpha^2 + 1 + D},$$

$$k_3' = \frac{1}{k_3 - 1} = \frac{\mu\alpha^2 - 1}{\mu\alpha^2 + 1}, \qquad k_3k_3' = \frac{2\mu\alpha^2}{\mu\alpha^2 + 1},$$

$$K' = k_3'\sqrt{k_1'k_2'}, \qquad \frac{1}{KK'} = \frac{\mu\alpha^2+1}{\sqrt{(\mu\alpha^2+1)^2 - D^2}} \qquad (46)$$

Shunda (44) ifodalar integrallanib, quyidagilar hosil boʻladi:

$$s(\theta) = s_0 \pm \frac{2\sqrt{\mu}}{KK'}\left(k_3k_3'I_2(\theta) - I_1(\theta)\right) \qquad (47)$$

Ya’ni

$$s(\theta) = s_0 \pm \frac{2\sqrt{\mu}}{\sqrt{(\mu\alpha^2+1)^2 - D^2}}\left(2\mu\alpha^2 I_2(\theta) - (\mu\alpha^2 + 1)I_1(\theta)\right) \qquad (48)$$

Va

$$x(\theta) = x_0 - s(\theta) + s_0 \pm \frac{4\sqrt{\mu}}{KK'}\left(k_3k_3'I_3(\theta) - I_2(\theta)\right) \qquad (49)$$

Bu yerda s_0 va x_0 — integrallash konstantalari, hamda

$$I_1(\theta) = \int\frac{dt}{\sqrt{P_4(t)}} = \sqrt{k_1 - 1}F(\varphi(\theta), k)$$

$$I_2(\theta) = \int\frac{dt}{(1+t^2)\sqrt{P_4(t)}} = \sqrt{k_1 - 1}\Pi(n, \varphi(\theta), k)$$

$$I_3(\theta) = \int\frac{dt}{(1+t^2)^2\sqrt{P_4(t)}} \qquad (50)$$

integrallar boʻlib, ular birinchi, ikkinchi va uchinchi turdagi toʻliq boʻlmagan elliptik integrallar orqali ifodalanadi. Uchinchi turdagi elliptik integral quyidagicha aniqlanadi:

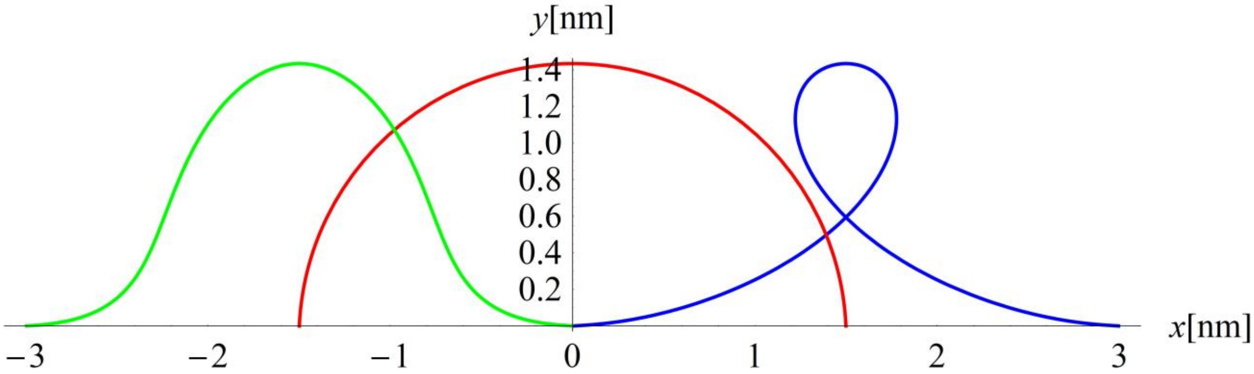
$$\Pi(n, \varphi, k) = \int_0^{\varphi}\frac{d\phi}{(1 - n\sin^2\phi)\sqrt{1 - k^2\sin^2\phi}} \qquad (51)$$

Bu yerda Jakobi amplitudasi $\phi(\theta)$, elliptik modul k va xarakteristik n quyidagicha beriladi:

$$\varphi(\theta) = \arcsin\left(\frac{tg\,\frac{\theta}{2}}{\sqrt{k_2'}}\right) = \arcsin\left(\sqrt{\frac{\mu\alpha^2 + 1 + D}{\mu\alpha^2 - 1 - D}}\,tg\,\frac{\theta}{2}\right)$$

$$k = \sqrt{\frac{k_2'}{k_1'}}, \qquad n = -k_2' \qquad (52)$$

Yuqoridagi oʻzgartirishlar va hosilalar yordamida, yakuniy natijada elastikaning koordinatalari toʻliq boʻlmagan elliptik integrallar orqali noaniq parametrik koʻrinishda ifodalanadi (2-rasmga qarang):



2-rasm: 58) va (42) bilan aniqlangan parametrik yechimning $(x(\theta),\ y(\theta))$ misoliy grafiklari, integratsiya konstantalari $x_0 = y_0 = 0$, $D = 0{,}1$ va nojoylama parametr $\mu = 2.25\ nm^2$, kuch $F = 500nN$, Yong moduli $E = 1000GPa$, inertiya momenti $I = 0.0491\ nm^4$, bu yerda yashil chiziq: $-\frac{\pi}{2} + \phi(\theta)$, qizil chiziq:$\phi(\theta)$, ko‘k chiziq: $\frac{\pi}{2} - \phi(\theta)$.

$$s(\theta) = s_0 \pm 2\sqrt{\mu}\frac{2\mu\alpha^2\Pi(n,\varphi(\theta),k) - (1+\mu\alpha^2)F(\varphi(\theta),k)}{\sqrt{(\mu\alpha^2 + D)^2 - 1}}$$

$$x(\theta) = x_0 - (y(\theta) - y_0)tg\frac{\theta}{2} \pm \frac{\sqrt{(\mu\alpha^2+D)^2-1}}{\sqrt{\mu}\alpha^2}E(\varphi(\theta),k) \mp \frac{2\sqrt{\mu}DF(\varphi(\theta),k)}{\sqrt{(\mu\alpha^2+D)^2-1}} \qquad (53)$$

bu yerda $y(\theta)$ (42)-ifoda orqali aniqlanadi.

Xulosa.

Mazkur tadqiqotda klassik Euler elastikasi va uning nolo-kal modifikatsiyasi uchun aniq analitik yechimlar olindi. Lokal modelda elastik chiziqning deformatsiyasi Yakobi elliptik funksiyalari orqali ifodalandi. Bu yechimlar egilish momenti, aksial kuch va geometrik parametrlarning o‘zaro ta’sirini aniq tasvirlash imkonini beradi. Nolo-kal modelni qo‘llash natijasida esa kichik o‘lchamli effektlarning elastik xatti-harakatga ta’siri aniqlashtirildi. Modifikatsiyalangan tenglama μ parametr orqali materialning ichki xarakteristik uzunligini hisobga oladi va μ → 0 holatda lokal modelga o‘tishi ko‘rsatildi. Olingan parametrik yechimlar birinchi, ikkinchi va uchinchi turdagi to‘liq bo‘lmagan elliptik integrallar yordamida ifodalandi. Bu esa nano-miqyosdagi elastik tizimlarning mexanik xossalarini chuqurroq o‘rganish imkonini beradi. Tadqiqot natijalari nano-balkalar, nanotubalar va boshqa kichik o‘lchamli konstruksiyalarning statik deformatsiyasini modellashtirishda muhim ahamiyat kasb etadi.

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